School of Mathematical and Physical Sciences

University of Reading

Error dynamics and instability in data assimilation

A.J.F. Moodey | A.S. Lawless | R.W.E. Potthast | P.J. van Leeuwen

Introduction

We consider the evolution of the analysis error in a cycled data assimilation system. We develop theory in a Hilbert space setting to show that stability of the error in time may be guaranteed by an appropriate inflation of the background error covariances.

Cycled data assimilation

At each assimilation step k, given measurements $y_k \in \mathbf{Y}$ with error covariance operator R and a prior estimate $x_k^{(b)} \in \mathbf{X}$ with error covariance operator B, where $\mathbf{X}$, $\mathbf{Y}$ are Hilbert spaces, minimize the cost function

$$\mathcal{J}(x_k^{(a)}) = \alpha \| x_k^{(a)} - x_k^{(b)} \|_{B^{-1}}^2 + \| y_k - H x_k^{(a)} \|_{R^{-1}}^2$$

to find the analysis $x_k^{(a)}$ where

$$x_k^{(b)} = \mathcal{M}_k(x_{k-1}^{(a)})$$

and H is a compact linear injective operator.

Decreasing the regularization parameter α acts to inflate the covariance matrix B.

Linear model dynamics

Theorem 1: Under the assumptions

- linear, time-invariant model, with additive error bounded by constant $\upsilon > 0$;
- imperfect observation operator, with error bounded by constant $\gamma > 0$;
- imperfect observations, with noise bounded by constant $\delta > 0$,

the norm of the analysis error $e_k = x_k^{(a)} - x_k^{(t)}$ evolves as

$$\|e_k\| \le \|\Lambda\|^k \|e_0\| + \sum_{l=0}^{k-1} \|\Lambda\|^l \left(\|N\| \upsilon + \|\mathcal{R}_\alpha\| (\delta + \gamma) \right)$$

with

$$\mathcal{R}_\alpha = (\alpha I + H^* H)^{-1} H^*, \; N = I - \mathcal{R}_\alpha H, \; \Lambda = NM$$

Corollary 2: If $\|\Lambda\| < 1$ then the analysis error is bounded for all time.

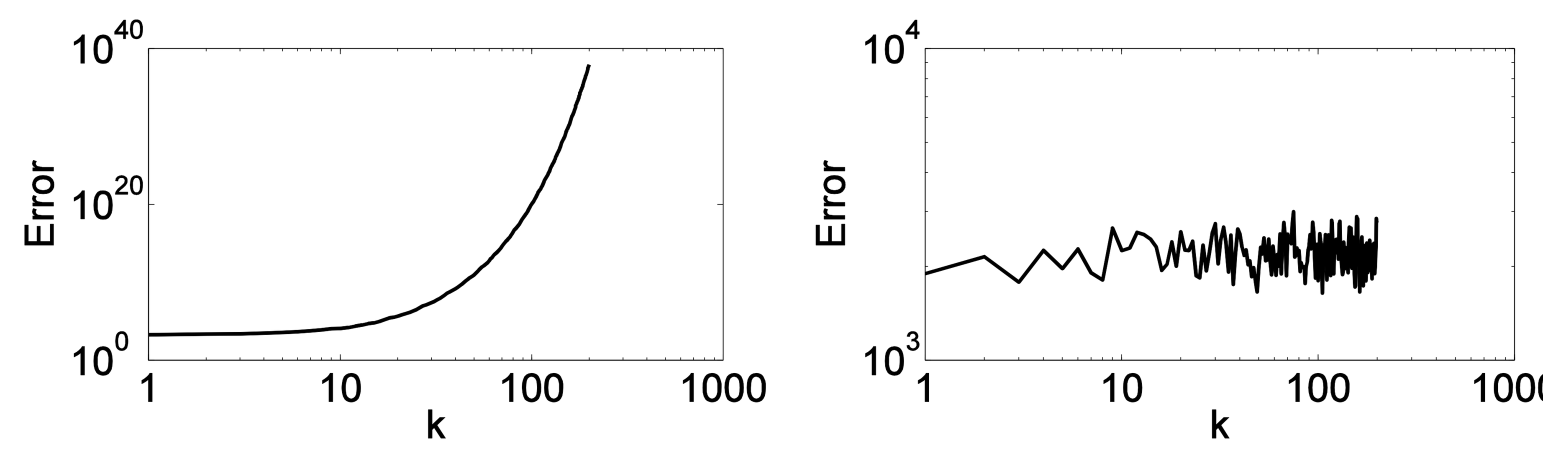
Theorem 3: Under the assumptions of Theorem 1, if M is a Hilbert Schmidt operator, then for regularization parameter $\alpha > 0$ sufficiently small we have $\|\Lambda\| < 1$ and the analysis error is bounded for all time by

$$\limsup_{k \to \infty} \|e_k\| \le \frac{\|N\| \upsilon + \|\mathcal{R}_\alpha\| (\delta + \gamma)}{1 - \|\Lambda\|} \le \frac{\upsilon + \frac{\delta + \gamma}{2\sqrt{\alpha}}}{1 - \|\Lambda\|}$$

- Whereas the stable inversion of each problem requires α to be sufficiently large, stability in time requires α to be sufficiently small (inflation of background error covariances).
- As α decreases, the magnitude of the bound on the error increases.
- The theory can be extended to time-varying dynamics by allowing α to vary with time.

Numerical illustration

Using the 2D Eady model of baroclinic instability we demonstrate the error growth with assimilation time k.



Norm of analysis error for α=16, determined from statistics (left) and for smaller value of α=1 (right)

- By choosing a smaller regularization parameter we can ensure that the cycled data assimilation remains stable.

Nonlinear model dynamics

Let $\mathcal{M}_k^{(2)}$ be the orthogonal projection of $\mathcal{M}_k$ into the subspace spanned by the higher spectral modes of H.

Definition 4: A nonlinear system $\mathcal{M}_k, \; k \in \mathbb{N}_0$ is *dissipative with respect to H* if it is Lipschitz continuous and damping with respect to the higher spectral modes of H, in the sense that

$$\left\| \mathcal{M}_k^{(2)}(\mathbf{x}_1) - \mathcal{M}_k^{(2)}(\mathbf{x}_2) \right\| \le K_k^{(2)} \cdot \|\mathbf{x}_1 - \mathbf{x}_2\|$$

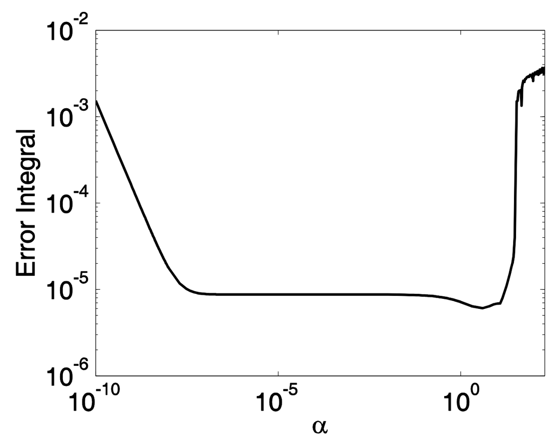
for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{X}$, where $K_k^{(2)} \le K^{(2)} < 1$ uniformly for $k \in \mathbb{N}_0$.

Theorem 5: If $\mathcal{M}_k$ is Lipschitz continuous and dissipative with respect to the higher spectral modes of H, then for regularization parameter $\alpha > 0$ sufficiently small the analysis error is bounded for all time.

- As for the linear problem we find the need to make α sufficiently small in order to ensure stability in time.
- The error bound has a very similar form to the linear case and increases as α decreases.

Numerical illustration

Using the stochastic Lorenz equations we show how the time-integral of the assimilation error over the first 1000 assimilation steps varies for different choices of the regularization parameter α.



Norm of time-averaged analysis error over 1000 steps for different values of α.

- If the regularization parameter is chosen to be too large then stability in time is not guaranteed.
- If the regularization parameter is too small then the assimilation problem at each time is ill-conditioned and again the error grows.

Conclusions

- New results for the stability of cycled data assimilation have been proved in a Hilbert space setting.
- The results show that stability can be guaranteed for the cases of
  - Linear dynamics with a Hilbert-Schmidt operator.
  - Nonlinear dynamics that are dissipative with respect to the observation operator H.
- Stability in time can be achieved by reducing the regularization parameter (or, equivalently, inflating the background error covariances). However, this makes each inversion problem more unstable.

References

1. Moodey, A.J.F., Lawless, A.S., Potthast, R.W.E. and van Leeuwen, P.J. (2013), **Nonlinear error dynamics for cycled data assimilation methods**, *Inverse Problems*, 29, 025002.
2. Moodey, A.J.F. (2013), **Instability and regularization for data assimilation**, PhD thesis, Dept of Mathematics and Statistics, University of Reading.
3. Potthast, R.W.E., Moodey, A.J.F., Lawless, A.S. and van Leeuwen, P.J. (2012), **On error dynamics and instability in data assimilation.**, Preprint MPS-2012-05, Dept of Mathematics and Statistics, University of Reading.

Acknowledgement

The Eady model code was developed at the University of Reading by Dr. C. Johnson, Prof N.K. Nichols and Prof B.J. Hoskins.

Contact information

- Department of Mathematics and Statistics, University of Reading, Whiteknights, RG6 6AX.
- Email: **a.s.lawless@reading.ac.uk** | **www.met.reading.ac.uk/~darc**

EPSRC Engineering and Physical Sciences Research Council

National Centre for Earth Observation NATURAL ENVIRONMENT RESEARCH COUNCIL