Stochastic Galerkin method for dynamic data assimilation using Wiener's polynomial chaos

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Let $X(t, \omega)$ be a real valued square integrable stochastic process. It is well known that this process can be represented using the so called polynomial chaos expansion developed by Wiener in 1938 and is given by $X(t, \omega) = \sum_{k=0}^{\infty} x_k(t) H_k(\xi)$ where ξ is the standard normal random variable, $H_k(x)$ is the Hermite polynomial of degree k in x. It is well known that $H_0(\xi) = 1$, for $k \geq 0$, $E[H_k(\xi)] = 0$, and $E[H_n(\xi)H_m(\xi)] = h_n \delta_{nm}$. That is, the Hermite polynomials are orthogonal with respect to the standard Gaussian weighting function.Let $X_N(t, \omega)$ be an approximation to $X(t, \omega)$ obtained by keeping only the first (N+1) terms of the infinite expansion. It can be shown that $X_N(t, \omega)$ converges (in mean square) to $X(t, \omega)$ as N $\rightarrow \infty$.

Let $X^f(t, \omega)$ be the solution of a stochastic dynamical system with (a)random initial condition, (b) random forcing, and (c) random parameters. Indeed, we can represent all the random variables and random processes including the solution using the (N+1) mode polynomial chaos expansion in the Hermite basis. Substituting this representation into the dynamical model and taking inner product with $H_k(\xi)$ for $0 \leq k \leq N$ and exploiting the orthogonality property of Hermite polynomials, we perform the standard Galerkin projection of the dynamics onto the (N+1) dimensional subspace. This results in a system of (N+1) systems of coupled ODE in the time varying coefficients $x_k^f(t)$. By solving this system, we can reconstruct $X_N^f(t, \omega) = \sum_{k=0}^{N} x_k^f(t) H_k(\xi)$. By drawing M independent samples $\xi(j)$ from the standard Gaussian distribution, we can obtain forecast ensemble $X_N^f(t, \omega)(j)$. Using this forecast ensemble, we can readily compute the forecast covaraiance matrix. Notice that, unlike the other contemporary methods, we solve the forecast dynamics only once and be able to generate the forecast ensemble rather easily by drawing samples from the standard normal distribution [1]. Let $Z(t) = x(t) + v(t)$ the observation with the known covaraince R. We can assimilate the observation either using (a) the stochastic framework with perturbed observations or (b) the deterministic ensemble transform methods to create the analysis ensemble. We illustrate the above framework using nonlinear model given by the stochastic differential equation $dx(t) = -4x(1 - x^2)dt + \sigma dw(t)$ where dwt is the standard Brownian incremental process with random initial conditions.

[1] D. Xiu (2010) Stochastic Computations, Princeton University Press